

Partial Randomness and Dimension of Recursively Enumerable Reals

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Abstract. A real α is called recursively enumerable (“r.e.” for short) if there exists a computable, increasing sequence of rationals which converges to α . It is known that the randomness of an r.e. real α can be characterized in various ways using each of the notions; program-size complexity, Martin-Löf test, Chaitin Ω number, the domination and Ω -likeness of α , the universality of a computable, increasing sequence of rationals which converges to α , and universal probability. In this paper, we generalize these characterizations of randomness over the notion of partial randomness by parameterizing each of the notions above by a real $T \in (0, 1]$, where the notion of partial randomness is a stronger representation of the compression rate by means of program-size complexity. As a result, we present ten equivalent characterizations of the partial randomness of an r.e. real. The resultant characterizations of partial randomness are powerful and have many important applications. One of them is to present equivalent characterizations of the dimension of an individual r.e. real. The equivalence between the notion of Hausdorff dimension and compression rate by program-size complexity (or partial randomness) has been established at present by a series of works of many researchers over the last two decades. We present ten equivalent characterizations of the dimension of an individual r.e. real.

Key words: algorithmic randomness, recursively enumerable real, partial randomness, dimension, Chaitin Ω number, program-size complexity, universal probability

1 Introduction

A real α is called *recursively enumerable* (“r.e.” for short) if there exists a computable, increasing sequence of rationals which converges to α . The randomness of an r.e. real α can be characterized in various ways using each of the notions; *program-size complexity*, *Martin-Löf test*, *Chaitin Ω number*, the *domination* and *Ω -likeness* of α , the *universality* of a computable, increasing sequence of rationals which converges to α , and *universal probability*. These equivalent characterizations of randomness of an r.e. real are summarized in Theorem 6 (see Section 3), where the equivalences are established by a series of works of Martin-Löf [9], Schnorr [14], Chaitin [4], Solovay [15], Calude, Hertling, Khoussainov and

Wang [1], Kučera and Slaman [7], and Tadaki [20], between 1966 and 2006. In this paper, we generalize these characterizations of randomness over the notion of *partial randomness*, which was introduced by Tadaki [18,19] and is a stronger representation of *the compression rate* by means of program-size complexity. We introduce many characterizations of partial randomness for an r.e. real by parameterizing each of the notions above on randomness by a real $T \in (0, 1]$. In particular, we introduce the notion of *T-convergence* for a computable, increasing sequence of rationals and then introduce the same notion for an r.e. real. The notion of *T-convergence* plays a crucial role in these our characterizations of partial randomness. We then prove the equivalence of all these characterizations of partial randomness in Theorem 8, our main result, in Section 4.

On the other hand, by a series of works of Ryabko [12,13], Staiger [16,17], Tadaki [18,19], Lutz [8], and Mayordomo [10] over the last two decades, the equivalence between the notion of compression rate by program-size complexity (or partial randomness) and Hausdorff dimension seems to be established at present. The subject of the equivalence seems to be one of the most active areas of the recent research of algorithmic randomness. In the context of the subject, we can consider the notion of the *dimension* of an individual real in particular, and this notion plays a crucial role in the subject. As one of the main applications of our main result on partial randomness, i.e., Theorem 8, we can present many equivalent characterizations of the dimension of an individual r.e. real.

The paper is organized as follows. We begin in Section 2 with some preliminaries to algorithmic information theory and partial randomness. In Section 3, we review the previous results on the equivalent characterizations of randomness of an r.e. real. Our main result on partial randomness of an r.e. real is presented in Section 4. In Section 5 we apply our main result on partial randomness to give many equivalent characterizations of the dimension of an r.e. real. In Section 6, we investigate further properties of the notion of *T-convergence*, which plays a crucial role in our characterizations of the partial randomness and dimension of r.e. reals. We conclude this paper with a mention of the future direction of this work in Section 7. Due to the 12-page limit, we omit most proofs. A full paper which describes all the proofs and other related results is in preparation.

2 Preliminaries

We start with some notation about numbers and strings which will be used in this paper. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and \mathbb{N}^+ is the set of positive integers. \mathbb{Q} is the set of rational numbers, and \mathbb{R} is the set of real numbers. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of numbers (rationals or reals) is called *increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

$\{0, 1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, \dots\}$ is the set of finite binary strings, where λ denotes the *empty string*. For any $s \in \{0, 1\}^*$, $|s|$ is the *length* of s . A subset S of $\{0, 1\}^*$ is called *prefix-free* if no string in S is a prefix of another string in S . For any partial function f , the domain of definition of f is denoted by $\text{dom } f$. We write “r.e.” instead of “recursively enumerable.”

Normally, $o(n)$ denotes any function $f: \mathbb{N}^+ \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any function $g: \mathbb{N}^+ \rightarrow \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \leq C$ for all $n \in \mathbb{N}^+$.

Let α be an arbitrary real. For any $n \in \mathbb{N}^+$, we denote by $\alpha \upharpoonright_n \in \{0,1\}^*$ the first n bits of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$ with infinitely many zeros, where $\lfloor \alpha \rfloor$ is the greatest integer less than or equal to α . Thus, in particular, if $\alpha \in [0, 1)$, then $\alpha \upharpoonright_n$ denotes the first n bits of the base-two expansion of α with infinitely many zeros. For example, in the case of $\alpha = 5/8$, $\alpha \upharpoonright_6 = 101000$.

A real α is called *r.e.* if there exists a computable, increasing sequence of rationals which converges to α . An r.e. real is also called a *left-computable* real. Let α and β be arbitrary r.e. reals. Then $\alpha + \beta$ is r.e. If α and β are non-negative, then $\alpha\beta$ is r.e. On the other hand, a real α is called *right-computable* if $-\alpha$ is left-computable. We say that a real α is *computable* if there exists a computable sequence $\{a_n\}_{n \in \mathbb{N}}$ of rationals such that $|\alpha - a_n| < 2^{-n}$ for all $n \in \mathbb{N}$. It is then easy to see that, for every $\alpha \in \mathbb{R}$, α is computable if and only if α is both left-computable and right-computable. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of reals is called *computable* if there exists a total recursive function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that $|a_n - f(n, m)| < 2^{-m}$ for all $n, m \in \mathbb{N}$. See e.g. Weihrauch [23] for the detail of the treatment of the computability of reals and sequences of reals.

2.1 Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [4,5]. A *computer* is a partial recursive function $C: \{0,1\}^* \rightarrow \{0,1\}^*$ such that $\text{dom } C$ is a prefix-free set. For each computer C and each $s \in \{0,1\}^*$, $H_C(s)$ is defined as $\min \{ |p| \mid p \in \{0,1\}^* \text{ \& } C(p) = s \}$ (may be ∞). A computer U is said to be *optimal* if for each computer C there exists $d \in \mathbb{N}$ with the following property; if $p \in \text{dom } C$, then there is $q \in \text{dom } U$ for which $U(q) = C(p)$ and $|q| \leq |p| + d$. It is easy to see that there exists an optimal computer. We choose a particular optimal computer U as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the *program-size complexity* of s or the *Kolmogorov complexity* of s . It follows that for every computer C there exists $d \in \mathbb{N}$ such that, for every $s \in \{0,1\}^*$, $H(s) \leq H_C(s) + d$.

For any optimal computer V , *Chaitin's halting probability* Ω_V of V is defined as $\sum_{p \in \text{dom } V} 2^{-|p|}$. The real Ω_V is also called *Chaitin Ω number*.

Definition 1 (weak Chaitin randomness, Chaitin [4,5]). *For any $\alpha \in \mathbb{R}$, we say that α is weakly Chaitin random if there exists $c \in \mathbb{N}$ such that $n - c \leq H(\alpha \upharpoonright_n)$ for all $n \in \mathbb{N}^+$.* \square

Chaitin [4] showed that, for every optimal computer V , Ω_V is weakly Chaitin random.

Definition 2 (Martin-Löf randomness, Martin-Löf [9]). *A subset C of $\mathbb{N}^+ \times \{0,1\}^*$ is called a Martin-Löf test if C is an r.e. set and*

$$\forall n \in \mathbb{N}^+ \quad \sum_{s \in C_n} 2^{-|s|} \leq 2^{-n},$$

where $\mathcal{C}_n = \{s \mid (n, s) \in \mathcal{C}\}$. For any $\alpha \in \mathbb{R}$, we say that α is Martin-Löf random if for every Martin-Löf test \mathcal{C} , there exists $n \in \mathbb{N}^+$ such that, for every $k \in \mathbb{N}^+$, $\alpha|_k \notin \mathcal{C}_n$. \square

Theorem 1 (Schnorr [14]). For every $\alpha \in \mathbb{R}$, α is weakly Chaitin random if and only if α is Martin-Löf random. \square

The program-size complexity $H(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $H(s)$ without referring to such a concept, i.e., as in the following, we first introduce a *universal probability* m , and then define $H(s)$ as $-\log_2 m(s)$. A universal probability is defined as follows [24].

Definition 3 (universal probability). A function $r: \{0, 1\}^* \rightarrow [0, 1]$ is called a lower-computable semi-measure if $\sum_{s \in \{0, 1\}^*} r(s) \leq 1$ and the set $\{(a, s) \in \mathbb{Q} \times \{0, 1\}^* \mid a < r(s)\}$ is r.e. We say that a lower-computable semi-measure m is a universal probability if for every lower-computable semi-measure r , there exists $c \in \mathbb{N}^+$ such that, for all $s \in \{0, 1\}^*$, $r(s) \leq cm(s)$. \square

The following theorem can be then shown (see e.g. Theorem 3.4 of Chaitin [4] for its proof).

Theorem 2. For every optimal computer V , the function $2^{-H_V(s)}$ of s is a universal probability. \square

By Theorem 2, we see that $H(s) = -\log_2 m(s) + O(1)$ for every universal probability m . Thus it is possible to define $H(s)$ as $-\log_2 m(s)$ with a particular universal probability m instead of as $H_U(s)$. Note that the difference up to an additive constant is nonessential to algorithmic information theory.

2.2 Partial randomness

In the works [18,19], we generalized the notion of the randomness of a real so that the degree of the randomness, which is often referred to as the partial randomness recently [2,11,3], can be characterized by a real T with $0 < T \leq 1$ as follows.

Definition 4 (weak Chaitin T -randomness). Let $T \in \mathbb{R}$ with $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that α is weakly Chaitin T -random if there exists $c \in \mathbb{N}$ such that $Tn - c \leq H(\alpha|_n)$ for all $n \in \mathbb{N}^+$. \square

Definition 5 (Martin-Löf T -randomness). Let $T \in \mathbb{R}$ with $T \geq 0$. A subset \mathcal{C} of $\mathbb{N}^+ \times \{0, 1\}^*$ is called a Martin-Löf T -test if \mathcal{C} is an r.e. set and

$$\forall n \in \mathbb{N}^+ \quad \sum_{s \in \mathcal{C}_n} 2^{-T|s|} \leq 2^{-n}.$$

For any $\alpha \in \mathbb{R}$, we say that α is Martin-Löf T -random if for every Martin-Löf T -test \mathcal{C} , there exists $n \in \mathbb{N}^+$ such that, for every $k \in \mathbb{N}^+$, $\alpha|_k \notin \mathcal{C}_n$. \square

In the case where $T = 1$, the weak Chaitin T -randomness and Martin-Löf T -randomness result in weak Chaitin randomness and Martin-Löf randomness, respectively. Tadaki [19] generalized Theorem 1 over the notion of T -randomness as follows.

Theorem 3 (Tadaki [19]). *Let T be a computable real with $T \geq 0$. Then, for every $\alpha \in \mathbb{R}$, α is weakly Chaitin T -random if and only if α is Martin-Löf T -random.* \square

Definition 6 (T -compressibility). *Let $T \in \mathbb{R}$ with $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that α is T -compressible if $H(\alpha \upharpoonright_n) \leq Tn + o(n)$, which is equivalent to $\limsup_{n \rightarrow \infty} H(\alpha \upharpoonright_n)/n \leq T$.* \square

For every $T \in [0, 1]$ and every $\alpha \in \mathbb{R}$, if α is weakly Chaitin T -random and T -compressible, then

$$\lim_{n \rightarrow \infty} \frac{H(\alpha \upharpoonright_n)}{n} = T. \quad (1)$$

The left-hand side of (1) is referred to as the *compression rate* of a real α in general. Note, however, that (1) does not necessarily imply that α is weakly Chaitin T -random. Thus, the notion of partial randomness is a stronger representation of compression rate.

In the works [18,19], we generalized Chaitin Ω number to $\Omega(T)$ as follows. For each optimal computer V and each real $T > 0$, the *generalized halting probability* $\Omega_V(T)$ of V is defined by

$$\Omega_V(T) = \sum_{p \in \text{dom } V} 2^{-\frac{|p|}{T}}.$$

Thus, $\Omega_V(1) = \Omega_V$. If $0 < T \leq 1$, then $\Omega_V(T)$ converges and $0 < \Omega_V(T) < 1$, since $\Omega_V(T) \leq \Omega_V < 1$. The following theorem holds for $\Omega_V(T)$.

Theorem 4 (Tadaki [18,19]). *Let V be an optimal computer and let $T \in \mathbb{R}$.*

- (i) *If $0 < T \leq 1$ and T is computable, then $\Omega_V(T)$ is weakly Chaitin T -random and T -compressible.*
- (ii) *If $1 < T$, then $\Omega_V(T)$ diverges to ∞ .* \square

3 Previous results on the randomness of an r.e. real

In this section, we review the previous results on the randomness of an r.e. real. First we review some notions on r.e. reals.

Definition 7 (Ω -likeness). *For any r.e. reals α and β , we say that α dominates β if there are computable, increasing sequences $\{a_n\}$ and $\{b_n\}$ of rationals and $c \in \mathbb{N}^+$ such that $\lim_{n \rightarrow \infty} a_n = \alpha$, $\lim_{n \rightarrow \infty} b_n = \beta$, and $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$. An r.e. real α is called Ω -like if it dominates all r.e. reals.* \square

Solovay [15] showed the following theorem. For its proof, see also Theorem 4.9 of [1].

Theorem 5 (Solovay [15]). *For every r.e. reals α and β , if α dominates β then $H(\beta \upharpoonright_n) \leq H(\alpha \upharpoonright_n) + O(1)$ for all $n \in \mathbb{N}^+$.* \square

Definition 8 (universality). *A computable, increasing and converging sequence $\{a_n\}$ of rationals is called universal if for every computable, increasing and converging sequence $\{b_n\}$ of rationals there exists $c \in \mathbb{N}^+$ such that $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$, where $\alpha = \lim_{n \rightarrow \infty} a_n$ and $\beta = \lim_{n \rightarrow \infty} b_n$.* \square

The previous results on the equivalent characterizations of randomness for an r.e. real are summarized in the following theorem.

Theorem 6 ([14,4,15,1,7,20]). *Let α be an r.e. real with $0 < \alpha < 1$. Then the following conditions are equivalent:*

- (i) *The real α is weakly Chaitin random.*
- (ii) *The real α is Martin-Löf random.*
- (iii) *The real α is Ω -like.*
- (iv) *For every r.e. real β , $H(\beta \upharpoonright_n) \leq H(\alpha \upharpoonright_n) + O(1)$ for all $n \in \mathbb{N}^+$.*
- (v) *There exists an optimal computer V such that $\alpha = \Omega_V$.*
- (vi) *There exists a universal probability m such that $\alpha = \sum_{s \in \{0,1\}^*} m(s)$.*
- (vii) *Every computable, increasing sequence of rationals which converges to α is universal.*
- (viii) *There exists a universal computable, increasing sequence of rationals which converges to α .* \square

The historical remark on the proofs of equivalences in Theorem 6 is as follows. Schnorr [14] showed that (i) and (ii) are equivalent to each other. Chaitin [4] showed that (v) implies (i). Solovay [15] showed that (v) implies (iii), (iii) implies (iv), and (iii) implies (i). Calude, Hertling, Khoussainov, and Wang [1] showed that (iii) implies (v), and (v) implies (vii). Kučera and Slaman [7] showed that (ii) implies (vii). Finally, (vi) was inserted in the course of the derivation from (v) to (viii) by Tadaki [20].

4 New results on the partial randomness of an r.e. real

In this section, we generalize Theorem 6 above over the notion of partial randomness. For that purpose, we first introduce some new notions. Let T be an arbitrary real with $0 < T \leq 1$ throughout the rest of this paper. These notions are parametrized by the real T .¹

Definition 9 (T -convergence). *An increasing sequence $\{a_n\}$ of reals is called T -convergent if $\sum_{n=0}^{\infty} (a_{n+1} - a_n)^T < \infty$. An r.e. real α is called T -convergent if there exists a T -convergent computable, increasing sequence of rationals which converges to α , i.e., if there exists an increasing sequence $\{a_n\}$ of rationals such that (i) $\{a_n\}$ is T -convergent, (ii) $\{a_n\}$ is computable, and (iii) $\lim_{n \rightarrow \infty} a_n = \alpha$.* \square

¹ The parameter T corresponds to the notion of “temperature” in the statistical mechanical interpretation of algorithmic information theory developed by Tadaki [21,22].

Note that every increasing and converging sequence of reals is 1-convergent, and thus every r.e. real is 1-convergent. In general, based on the following lemma, we can freely switch from “ T -convergent computable, increasing sequence of reals” to “ T -convergent computable, increasing sequence of rationals.”

Lemma 1. *For every $\alpha \in \mathbb{R}$, α is an r.e. T -convergent real if and only if there exists a T -convergent computable, increasing sequence of reals which converges to α .* \square

The following argument illustrates the way of using Lemma 1: Let V be an optimal computer, and let p_0, p_1, p_2, \dots be a recursive enumeration of the r.e. set $\text{dom } V$. Then $\Omega_V(T) = \sum_{i=0}^{\infty} 2^{-|p_i|/T}$, and the increasing sequence $\{\sum_{i=0}^n 2^{-|p_i|/T}\}_{n \in \mathbb{N}}$ of reals is T -convergent since $\Omega_V = \sum_{i=0}^{\infty} 2^{-|p_i|} < 1$. If T is computable, then this sequence of reals is computable. Thus, by Lemma 1 we have Theorem 7 below.

Theorem 7. *Let V be an optimal computer. If T is computable, then $\Omega_V(T)$ is an r.e. T -convergent real.* \square

Definition 10 ($\Omega(T)$ -likeness). *An r.e. real α is called $\Omega(T)$ -like if it dominates all r.e. T -convergent reals.* \square

Note that an r.e. real α is $\Omega(1)$ -like if and only if α is Ω -like.

Definition 11 (T -universality). *A computable, increasing and converging sequence $\{a_n\}$ of rationals is called T -universal if for every T -convergent computable, increasing and converging sequence $\{b_n\}$ of rationals there exists $c \in \mathbb{N}^+$ such that $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$, where $\alpha = \lim_{n \rightarrow \infty} a_n$ and $\beta = \lim_{n \rightarrow \infty} b_n$.* \square

Note that a computable, increasing and converging sequence $\{a_n\}$ of rationals is 1-universal if and only if $\{a_n\}$ is universal.

Using the notions introduced above, Theorem 6 is generalized as follows.

Theorem 8 (main result). *Let α be an r.e. real with $0 < \alpha < 1$. Suppose that T is computable. Then the following conditions are equivalent:*

- (i) *The real α is weakly Chaitin T -random.*
- (ii) *The real α is Martin-Löf T -random.*
- (iii) *The real α is $\Omega(T)$ -like.*
- (iv) *For every r.e. T -convergent real β , $H(\beta \upharpoonright_n) \leq H(\alpha \upharpoonright_n) + O(1)$ for all $n \in \mathbb{N}^+$.*
- (v) *For every r.e. T -convergent real $\gamma > 0$, there exist an r.e. real $\beta \geq 0$ and a rational $q > 0$ such that $\alpha = \beta + q\gamma$.*
- (vi) *For every optimal computer V , there exist an r.e. real $\beta \geq 0$ and a rational $q > 0$ such that $\alpha = \beta + q\Omega_V(T)$.*
- (vii) *There exist an optimal computer V and an r.e. real $\beta \geq 0$ such that $\alpha = \beta + \Omega_V(T)$.*
- (viii) *There exists a universal probability m such that $\alpha = \sum_{s \in \{0,1\}^*} m(s)^{\frac{1}{T}}$.*

- (ix) Every computable, increasing sequence of rationals which converges to α is T -universal.
- (x) There exists a T -universal computable, increasing sequence of rationals which converges to α . \square

We see that Theorem 8 is a massive expansion of Theorem 3 in the case where the real α is r.e. with $0 < \alpha < 1$. The condition (vii) of Theorem 8 corresponds to the condition (v) of Theorem 6. Note, however, that, in the condition (vii) of Theorem 8, a non-negative r.e. real β is needed. The reason is as follows: In the case of $\beta = 0$, the possibility that α is weakly Chaitin T' -random with a real $T' > T$ is excluded by the T -compressibility of $\Omega_V(T)$ imposed by Theorem 4 (i). However, this exclusion is inconsistent with the condition (i) of Theorem 8.

Theorem 8 can be proved by generalizing the proof of Theorem 6 over the notion of partial randomness. For example, using Lemma 2 below, the implication (ii) \Rightarrow (v) of Theorem 8 is proved as follows, in which the notion of T -convergence plays an important role.

Proof (of (ii) \Rightarrow (v) of Theorem 8). Suppose that γ is an arbitrary r.e. T -convergent real with $\gamma > 0$. Then there exists a T -convergent computable, increasing sequence $\{c_n\}$ of rationals which converges to γ . Since $\gamma > 0$, without loss of generality we can assume that $c_0 = 0$. We choose any one rational $\varepsilon > 0$ such that $\sum_{n=0}^{\infty} [\varepsilon(c_{n+1} - c_n)]^T \leq 1$. Such ε exists since the sequence $\{c_n\}$ is T -convergent. Note that the sequence $\{\varepsilon(c_{n+1} - c_n)\}$ is a computable sequence of positive rationals. Thus, since α is a positive r.e. real and also Martin-Löf T -random by the assumption, it follows from Lemma 2 below that there exist a computable, increasing sequence $\{a_n\}$ of rationals and a rational $r > 0$ such that $a_{n+1} - a_n > r\varepsilon(c_{n+1} - c_n)$ for every $n \in \mathbb{N}$, $a_0 > 0$, and $\alpha = \lim_{n \rightarrow \infty} a_n$. We then define a sequence $\{b_n\}$ of positive rationals by $b_n = a_{n+1} - a_n - r\varepsilon(c_{n+1} - c_n)$. It follows that $\{b_n\}$ is a computable sequence of rationals and $\sum_{n=0}^{\infty} b_n$ converges to $\alpha - a_0 - r\varepsilon(\gamma - c_0)$. Thus we have $\alpha = a_0 + \sum_{n=0}^{\infty} b_n + r\varepsilon\gamma$, where $a_0 + \sum_{n=0}^{\infty} b_n$ is a positive r.e. real. This completes the proof. \square

Lemma 2. *Let α be an r.e. real, and let $\{d_n\}$ be a computable sequence of positive rationals such that $\sum_{n=0}^{\infty} d_n^T \leq 1$. If α is Martin-Löf T -random, then for every $\varepsilon > 0$ there exist a computable, increasing sequence $\{a_n\}$ of rationals and a rational $q > 0$ such that $a_{n+1} - a_n > qd_n$ for every $n \in \mathbb{N}$, $a_0 > \alpha - \varepsilon$, and $\alpha = \lim_{n \rightarrow \infty} a_n$. \square*

Lemma 2 can be proved, based on the generalization of the techniques used in the proof of Theorem 2.1 of Kučera and Slaman [7] over partial randomness. In addition to the proof of Lemma 2, the complete proof of Theorem 8 will be described in a full version of this paper, which is in preparation.

Theorem 8 has many important applications. One of the main applications is to give many characterizations of the dimension of an individual r.e. real, some of which will be presented in the next section. As another consequence of Theorem 8, we can obtain Corollary 1 below for example, which follows immediately from the implication (vii) \Rightarrow (iv) of Theorem 8 and Theorem 7.

Corollary 1. *Suppose that T is computable. Then, for every two optimal computers V and W , $H(\Omega_V(T)|_n) = H(\Omega_W(T)|_n) + O(1)$ for all $n \in \mathbb{N}^+$. \square*

Note that the computability of T is important for Theorem 8 to hold. For example, we cannot allow T to be simply an r.e. real in Theorem 8.

The notion of T -convergence has many interesting properties, in addition to the properties which we saw above. In Section 6, we investigate further properties of the notion of T -convergence.

5 New characterizations of the dimension of an r.e. real

In this section we apply Theorem 8 to give many characterizations of dimension for an individual r.e. real. In the works [18,19], we introduced the notions of six “algorithmic dimensions”, 1st, 2nd, 3rd, 4th, upper, and lower algorithmic dimensions as fractal dimensions for a subset F of N -dimensional Euclidean space \mathbb{R}^N . These notions are defined based on the notion of partial randomness and compression rate by means of program-size complexity. We then showed that all the six algorithmic dimensions equal to the Hausdorff dimension for any self-similar set which is computable in a certain sense. The class of such self-similar sets includes familiar fractal sets such as the Cantor set, von Koch curve, and Sierpiński gasket. In particular, the notion of lower algorithmic dimension for a subset F of \mathbb{R} is defined as follows.

Definition 12 (lower algorithmic dimension, Tadaki [19]). *Let F be a nonempty subset of \mathbb{R} . The lower algorithmic dimension $\underline{\dim}_A F$ of F is defined by $\underline{\dim}_A F = \sup_{x \in F} \liminf_{n \rightarrow \infty} H(x|_n)/n$. \square*

Thus, for every $\alpha \in \mathbb{R}$,

$$\underline{\dim}_A \{\alpha\} = \liminf_{n \rightarrow \infty} \frac{H(\alpha|_n)}{n}. \quad (2)$$

Independently of us, Lutz [8] introduced the notion of constructive dimension of an individual real α using the notion of lower semicomputable s -supergale with $s \in [0, \infty)$, and then Mayordomo [10] showed that, for every real α , the constructive dimension of α equals to the right-hand side of (2). Thus, the constructive dimension of α is precisely the lower algorithmic dimension $\underline{\dim}_A \{\alpha\}$ of α for every real α .

Using Lemma 3 below, we can convert each of all the conditions in Theorem 8 into a characterization of the lower algorithmic dimension $\underline{\dim}_A \{\alpha\}$ for any r.e. real α .

Lemma 3. *Let $\alpha \in \mathbb{R}$. For every $t \in [0, \infty)$, α is weakly Chaitin t -random if $t < \underline{\dim}_A \{\alpha\}$, and α is not weakly Chaitin t -random if $t > \underline{\dim}_A \{\alpha\}$.*

Proof. Let $\alpha \in \mathbb{R}$, and let $t \in [0, \infty)$. Assume first that $t < \underline{\dim}_A \{\alpha\}$. Then, since $\underline{\dim}_A \{\alpha\}n \leq H(\alpha|_n) + o(n)$ for all $n \in \mathbb{N}^+$, we see that

$$tn + \left(\underline{\dim}_A \{\alpha\} - t - \frac{o(n)}{n} \right) n \leq \underline{\dim}_A \{\alpha\}n - o(n) \leq H(\alpha|_n)$$

for all $n \in \mathbb{N}^+$. Thus, since $\underline{\dim}_A\{\alpha\} - t - o(n)/n > 0$ for all sufficiently large n , we see that α is weakly Chaitin t -random.

On the other hand, assume that α is weakly Chaitin t -random. Then we see that $t \leq \liminf_{n \rightarrow \infty} H(\alpha|_n)/n = \underline{\dim}_A\{\alpha\}$. Thus, if $t > \underline{\dim}_A\{\alpha\}$ then α is not weakly Chaitin t -random. This completes the proof. \square

For example, using Lemma 3, the condition (iii) in Theorem 8 is converted as follows. In this paper, we interpret the supremum $\sup \emptyset$ of the empty set as 0.

Theorem 9. *Let α be an r.e. real. Then, for every $t \in (0, 1]$, α is $\Omega(t)$ -like if $t < \underline{\dim}_A\{\alpha\}$, and α is not $\Omega(t)$ -like if $t > \underline{\dim}_A\{\alpha\}$. Thus,*

$$\underline{\dim}_A\{\alpha\} = \sup\{t \in (0, 1] \mid \alpha \text{ is } \Omega(t)\text{-like}\}. \quad \square$$

On the other hand, the condition (viii) in Theorem 8 is converted as follows, using Lemma 3. Here \mathbb{R}_c denotes the set of all computable reals.

Theorem 10. *Let α be an r.e. real with $0 < \alpha < 1$. Then, for every $t \in (0, 1] \cap \mathbb{R}_c$, if $t < \underline{\dim}_A\{\alpha\}$ then $\alpha = \sum_{s \in \{0,1\}^*} m(s)^{\frac{1}{t}}$ for some universal probability m , and if $t > \underline{\dim}_A\{\alpha\}$ then $\alpha \neq \sum_{s \in \{0,1\}^*} m(s)^{\frac{1}{t}}$ for any universal probability m . Thus, $\underline{\dim}_A\{\alpha\} = \sup S$, where S is the set of all $t \in (0, 1] \cap \mathbb{R}_c$ such that $\alpha = \sum_{s \in \{0,1\}^*} m(s)^{\frac{1}{t}}$ for some universal probability m . \square*

In the same manner, using Lemma 3 we can convert each of the remaining eight conditions in Theorem 8 also into a characterization of the lower algorithmic dimension of an r.e. real. In a full version of this paper, we will describe the complete list of the ten characterizations of the lower algorithmic dimension obtained from Theorem 8.

6 Further properties of T -convergence

In this section, we investigate further properties of the notion of T -convergence. First, as one of the applications of Theorem 8, the following theorem can be obtained.

Theorem 11. *Suppose that T is computable. For every r.e. real α , if α is T -convergent, then α is T -compressible.*

Proof. Using (vii) \Rightarrow (iv) of Theorem 8, we see that, for every r.e. T -convergent real α , $H(\alpha|_n) \leq H(\Omega_U(T)|_n) + O(1)$ for all $n \in \mathbb{N}^+$. It follows from Theorem 4 (i) that α is T -compressible for every r.e. T -convergent real α . \square

In the case of $T < 1$, the converse of Theorem 11 does not hold, as seen in Theorem 12 below in a sharper form. Theorem 12 can be proved partly using (vii) \Rightarrow (ix) of Theorem 8.

Theorem 12. *Suppose that T is computable and $T < 1$. Then there exists an r.e. real η such that (i) η is weakly Chaitin T -random and T -compressible, and (ii) η is not T -convergent. \square*

Let T_1 and T_2 be arbitrary computable reals with $0 < T_1 < T_2 < 1$, and let V be an arbitrary optimal computer. By Theorem 4 (i) and Theorem 11, we see that the r.e. real $\Omega_V(T_2)$ is not T_1 -convergent and therefore every computable, increasing sequence $\{a_n\}$ of rationals which converges to $\Omega_V(T_2)$ is not T_1 -convergent. At this point, conversely, the following question arises naturally: Is there any computable, increasing sequence of rationals which converges to $\Omega_V(T_1)$ and which is not T_2 -convergent? We can answer this question affirmatively in the form of Theorem 13 below.

Theorem 13. *Let T_1 and T_2 be arbitrary computable reals with $0 < T_1 < T_2 < 1$. Then there exist an optimal computer V and a computable, increasing sequence $\{a_n\}$ of rationals such that (i) $\Omega_V(T_1) = \lim_{n \rightarrow \infty} a_n$, (ii) $\{a_n\}$ is T -convergent for every $T \in (T_2, \infty)$, and (iii) $\{a_n\}$ is not T -convergent for every $T \in (0, T_2]$.* \square

7 Concluding remarks

In this paper, we have generalized the equivalent characterizations of randomness of a recursively enumerable real over the notion of partial randomness, so that the generalized characterizations are all equivalent to the weak Chaitin T -randomness. As a stronger notion of partial randomness of a real α , Tadaki [18,19] introduced the notion of the Chaitin T -randomness of α , which is defined as the condition on α that $\lim_{n \rightarrow \infty} H(\alpha|_n) - Tn = \infty$.² Thus, future work may aim at modifying our equivalent characterizations of partial randomness so that they become equivalent to the Chaitin T -randomness.

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² The actual separation of the Chaitin T -randomness from the weak Chaitin T -randomness is done by Reimann and Stephan [11].

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